GAUSS-JORDAN METHOD OF SOLVING SYSTEMS OF EQUATIONS THEORY

We'll start with 2 equations and 2 unknowns:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

Step 1:

Write the augmented matrix and label the rows:

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Step 2:

Obtain a 1 where a_{11} is. This requires a multiplication or division step; or, if there is a 1 directly below a_{11}, an exchange of rows. So, multiply all the entries in row A by $1/a_{11}$ or divide all the entries in row A by a_{11}.

$$\begin{array}{c} A \\ \\ B \end{array} \left[\begin{array}{cc|c} 1 & \dfrac{a_{12}}{a_{11}} & \dfrac{b_1}{a_{11}} \\ \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Now, our matrix is looking pretty ugly, so we'll improve its looks by re-labeling the entries:

$$\begin{array}{c} A \\ \\ B \end{array} \left[\begin{array}{cc|c} 1 & c_{12} & d_1 \\ \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

Step 3:

We now need to get a 0 where a_{21} is. This requires an addition or subtraction step (similar to our foolproof method). So, working column by column, replace row B with

(the entry from row B) - (a_{21}) (the entry from row A)

$$\begin{array}{l} A \\ B \end{array} \left[\begin{array}{ccc|c} 1 & & c_{12} & d_1 \\ (a_{21}) - (a_{21})(1) & & (a_{22}) - (a_{21})(c_{12}) & (b_2) - (a_{21})(d_1) \end{array}\right]$$

Make this pretty.

$$\begin{array}{l} A \\ B \end{array} \left[\begin{array}{ccc|c} 1 & & c_{12} & d_1 \\ 0 & & c_{22} & d_2 \end{array}\right]$$

Step 4:

We now need to get a 1 where c_{22} is. This requires a multiplication or division step. So, multiply all the entries in row B by $1/c_{22}$ or divide all the entries in row B by c_{22}.

$$\begin{array}{l} A \\ B \end{array} \left[\begin{array}{ccc|c} 1 & & c_{12} & d_1 \\ 0 & & 1 & \dfrac{d_2}{c_{22}} \end{array}\right]$$

Make this pretty.

$$\begin{array}{l} A \\ B \end{array} \left[\begin{array}{ccc|c} 1 & & c_{12} & d_1 \\ 0 & & 1 & f_2 \end{array}\right]$$

Step 5:

We now need to get a 0 where c_{12} is. We must not mess up the first column. This requires an addition or subtraction step. So, working column by column, replace row A by

(the entry from row A) - (c_{12}) (the entry from row B)

$$\begin{array}{cc} A \\ B \end{array} \left[\begin{array}{ccc} (1) - (c_{12})(0) & (c_{12}) - (c_{12})(1) & (d_1) - (c_{12})(f_2) \\ 0 & 1 & f_2 \end{array}\right]$$

Make this pretty.

$$\begin{array}{cc} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 0 & f_1 \\ 0 & 1 & f_2 \end{array}\right]$$

Step 6:
Interpret the matrix.

$x = f_1$
$y = f_2$

EXAMPLE 1

Solve: $3x + 2y = 13$
$4x + 3y = 18$

Step 1:
Write the augmented matrix:

$$\begin{array}{cc} A \\ B \end{array} \left[\begin{array}{cc|c} 3 & 2 & 13 \\ 4 & 3 & 18 \end{array}\right]$$

Step 2:
Divide the entries in row A by 3.

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 2/3 & 13/3 \\ 4 & 3 & 18 \end{array}\right]$$

Step 3:
Replace row B by (the entry from row B) - (4) (the entry from row A)

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 2/3 & 13/3 \\ (4) - (4)(1) & (3) - (4)(2/3) & (18) - (4)(13/3) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 2/3 & 13/3 \\ 0 & 1/3 & 2/3 \end{array}\right]$$

Step 4:
Multiply all the entries in row B by 3.

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 2/3 & 13/3 \\ 0 & 1 & 2 \end{array}\right]$$

Step 5:
Replace row A by (the entry from row A) - (2/3) (the entry from Row B)

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} (1) - (2/3)(0) & (2/3) - (2/3)(1) & (13/3) - (2/3)(2) \\ 0 & 1 & 2 \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

Step 6:
Interpret the matrix.

$x = 3$
$y = 2$

<u>NOTE</u>: IN THE REMAINING EXAMPLES THE STATEMENT

(the entry from row A) - (2) (the entry from row B)

WILL BE SHORTENED TO

A - 2B

ALL SIMILAR STATEMENTS WILL BE SIMILARLY SHORTENED

<u>EXAMPLE 2</u>

Solve: $5x - 2y = 16$
$4x + 3y = -1$

Step 1:
Make the matrix

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 5 & -2 & 16 \\ 4 & 3 & -1 \end{array}\right]$$

Step 2:
Multiply each entry in row A by 1/5

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & -2/5 & 16/5 \\ 4 & 3 & -1 \end{array}\right]$$

Step 3:
Column by column, replace row B by B - 4A

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & -2/5 & 16/5 \\ (4) - (4)(1) & (3) - (4)(-2/5) & (-1) - (4)(16/5) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & -2/5 & 16/5 \\ 0 & 23/5 & -69/5 \end{array}\right]$$

Step 4:
Multiply each entry in row B by 5/23

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} 1 & -2/5 & 16/5 \\ 0 & 1 & -3 \end{array}\right]$$

Step 5:
Column by column, replace row A by A – (-2/5) B or A + (2/5) B

$$\begin{array}{c} A \\ B \end{array} \left[\begin{array}{cc|c} (1) + (2/5)(0) & (-2/5) + (2/5)(1) & (16/5) + (2/5)(-3) \\ 0 & 1 & -3 \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} \\ A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array}\right]$$

Step 6:
Interpret the matrix.

$x = 2$
$y = -3$

EXAMPLE 3

Solve:

$$\begin{aligned} 7x - 5y &= -70 \\ x + 3y &= 16 \end{aligned}$$

Step 1:
Write the matrix.

$$\begin{array}{c} \\ A \\ B \end{array} \left[\begin{array}{cc|c} 7 & -5 & -70 \\ 1 & 3 & 16 \end{array}\right]$$

Step 2:
Exchange rows A and B.

$$\begin{array}{c} \\ A \\ B \end{array} \left[\begin{array}{cc|c} 1 & 3 & 16 \\ 7 & -5 & -70 \end{array}\right]$$

Step 3:
Column by column, replace row B by B - 7A

$$\begin{array}{cc} A & \\ B & \end{array}\left[\begin{array}{ccc} 1 & 3 & 16 \\ (7) - (7)(1) & (-5) - (7)(3) & (-70) - (7)(16) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{cc} A & \\ B & \end{array}\left[\begin{array}{ccc} 1 & 3 & 16 \\ 0 & -26 & -182 \end{array}\right]$$

Step 4:
Multiply each entry in row B by -1/26

$$\begin{array}{cc} A & \\ B & \end{array}\left[\begin{array}{ccc} 1 & 3 & 16 \\ 0 & 1 & 7 \end{array}\right]$$

Step 5:
Column by column, replace row A by A - 3B

$$\begin{array}{cc} A & \\ B & \end{array}\left[\begin{array}{ccc} (1) - (3)(0) & (3) - (3)(1) & (16) - (3)(7) \\ 0 & 1 & 7 \end{array}\right]$$

Perform the calculations.

$$\begin{array}{cc} A & \\ B & \end{array}\left[\begin{array}{ccc} 1 & 0 & -5 \\ 0 & 1 & 7 \end{array}\right]$$

Step 6:
Interpret the matrix.

$x = -5$
$y = 7$

<u>NOTE</u>: **EXAMPLES WITH 3 EQUATIONS AND 3 UNKNOWNS WILL REQUIRE 2 EXTRA STEPS. PAY CLOSE ATTENTION.**

<u>EXAMPLE 4</u>

Solve: $x + 5y + 7z = 20$

$2x - 3y + 4z = 4$

$3x + y - 2z = 9$

Step 1:
Write the matix.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 20 \\ 7 & -3 & 4 & 4 \\ 3 & 1 & -2 & 9 \end{array}\right]$$

Step 2:
It is already accomplished since we have a 1 in the first column of row A.

Step 3:
To get zeros in the first column of rows B and C, do the following replacements

Row B is replaced by B - 2A
and
Row C is replaced by C - 3A

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 20 \\ (2)-(2)(1) & (-3)-(2)(5) & (4)-(2)(7) & (4)-(2)(20) \\ (3)-(3)(1) & (1)-(3)(5) & (-2)-(3)(7) & (9)-(3)(20) \end{array}\right]$$

Perform the calculations

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 20 \\ 0 & -13 & -10 & -36 \\ 0 & -1 & -23 & -51 \end{array}\right]$$

Step 4:
To get a 1 in the second column of row B, multiply each entry in row B by -1/13.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 20 \\ 0 & 1 & 10/13 & 36/13 \\ 0 & -14 & -23 & -51 \end{array}\right]$$

Step 5:
To get zeros in the second column of row A and C, do the following replacements

Row A is replaced by A - 5B
and
Row C is replaced by C - (-14) B = C + 14B

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} (1) - (5)(0) & (5) - (5)(1) & (7) - (5)(10/13) & (20) - (5)(36/13) \\ 0 & 1 & 10/13 & 36/13 \\ (0) + (14)(0) & (-14) + (14)(1) & (-23) + (14)(10/13) & (-51) + (14)(36/13) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 41/13 & 80/13 \\ 0 & 1 & 10/13 & 36/13 \\ 0 & 0 & -159/13 & -159/13 \end{array}\right]$$

Step 6:
To get a 1 in the third column of row C, multiply row C by -13/159

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 41/13 & 80/13 \\ 0 & 1 & 10/13 & 36/13 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Step 7:
To get zeros in the third column of rows A and B, do the following replacements

Row A by A - (41/13) C
and
Row B by B - (10/13) C

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} (1) - (41/13)(0) & (0) - (41/13)(0) & (41/13) - (41/13)(1) & (80/13) - (41/13)(1) \\ (0) - (10/13)(0) & (1) - (10/13)(0) & (10/13) - (10/13)(1) & (36/13) - (10/13)(1) \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Step 8:
Interpret the matrix.

$x = 3$

$y = 2$

$z = 1$

EXAMPLE 5

Solve:

$$3x + 7y - 2z = 2$$
$$x - 5y + z = 13$$
$$2x + 3y - 10z = -23$$

Step 1:
Write the matrix.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 3 & 7 & -2 & 2 \\ 1 & -5 & 1 & 13 \\ 2 & 3 & -10 & -23 \end{array}\right]$$

Step 2:
Exchange rows A and B.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 13 \\ 3 & 7 & -2 & 2 \\ 2 & 3 & -10 & -23 \end{array}\right]$$

Step 3:
Replace row B by B - 3A
and
Row C by C - 2A

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 13 \\ (3) - (3)(1) & (7) - (3)(-5) & (-2) - (3)(1) & (2) - (3)(13) \\ (2) - (2)(1) & (3) - (2)(-5) & (-10) - (2)(1) & (-23) - (2)(13) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & -5 & 7 & 13 \\ 0 & 22 & -5 & -37 \\ 0 & 13 & -12 & -49 \end{array}\right]$$

Step 4:
Multiply row B by 1/22

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 13 \\ 0 & 1 & -5/22 & -37/22 \\ 0 & 13 & -12 & -49 \end{array}\right]$$

Step 5:
Replace row A by A - (-5) B = A + 5B
and
Row C by C - 13B

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} (1) + (5)(0) & (-5) + (5)(1) & (1) + (5)(-5/22) & (13) + (5)(-37/22) \\ 0 & 1 & -5/22 & -37/22 \\ (0) - (13)(0) & (13) - (13)(1) & (-12) - (13)(-5/22) & (-49) - (13)(37/22) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3/22 & 101/22 \\ 0 & 1 & -5/22 & -37/22 \\ 0 & 0 & -199/22 & -597/22 \end{array}\right]$$

Step 6:
Multiply row C by -22/199

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{cccc} 1 & 0 & -3/22 & 101/22 \\ 0 & 1 & -5/22 & -37/22 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 7:
Replace row A by A - (-3/22) C = A + (3/22) C
and
Row B by B - (-5/22)C = B + (5/22)C

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{cccc} (1) + (3/22)(0) & (0) + (3/22)(0) & (-3/22) + (3/22)(1) & (101/22) + (3/22)(3) \\ (0) + (5/22)(0) & (1) + (5/22)(0) & (-5/22) + (5/22)(1) & (-37/22) + (5/22)(3) \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{cccc} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Step 8:
Interpret the matrix

$x = 5$

$y = -1$

$z = 3$

EXAMPLE 6

Solve:

$$x + 3y - 7z = -51$$

$$2x + y + 4z = 5$$

$$3x - 4y + z = 26$$

Step 1:
Write the matrix.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -7 & -51 \\ 2 & 1 & 4 & 5 \\ 3 & -4 & 1 & 26 \end{array}\right]$$

Step 2:
Not necessary in this problem

Step 3:
Replace row B by B - 2A
and
Row C by C - 3A

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -7 & -51 \\ (2) - (2)(1) & (1) - (2)(3) & (4) - (2)(-7) & (5) - (2)(-51) \\ (3) - (3)(1) & (-4) - (3)(3) & (1) - (3)(-7) & (26) - (3)(-51) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -7 & -51 \\ 0 & -5 & 18 & 107 \\ 0 & -13 & 22 & 179 \end{array}\right]$$

Step 4:
Multiply row B by -1/5

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 3 & -7 & -51 \\ 0 & 1 & -18/5 & -107/5 \\ 0 & -13 & 22 & 179 \end{array}\right]$$

Step 5:
Replace row A by A - 3B
and
Row C by C - (-13)B = C + 13B

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} (1) - (3)(0) & (3) - (3)(1) & (-7) - (3)(-18/5) & (-51) - (3)(-107/5) \\ 0 & 1 & -18/5 & -107/5 \\ (0) + (13)(0) & (-13) + (13)(1) & (22) + (13)(-18/5) & (179) + (13)(-107/5) \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 19/5 & 66/5 \\ 0 & 1 & -18/5 & -107/5 \\ 0 & 0 & -124/5 & -496/5 \end{array}\right]$$

Step 6:
Multiply row C by -5/124

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 19/5 & 66/5 \\ 0 & 1 & -18/5 & -107/5 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Step 7:
Replaced row A by A - (19/5)C
and
Row B by B - (-18/5)C = B + (18/5) C

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} (1) - (19/5)(0) & (0) - (19/5)(0) & (19/5) - (19/5)(1) & (66/5) - (19/5)(4) \\ (0) + (18/5)(0) & (1) + (18/5)(0) & (-18/5) + (18/5)(1) & (-107/5) + (18/5)(4) \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Perform the calculations.

$$\begin{array}{c} A \\ B \\ C \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Step 8:
Interpret the matrix

$x = -2$

$y = -7$

$z = 4$

THAT'S ALL, FOLKS.